Mean-field SDEs driven by supercritical α-stable process: wellposedness, PoC, and Euler approximation

Zimo Hao 郝子墨

Zimohao.github.io

Bielefeld University

Joint work with Chongyang Ren and Mingyan Wu

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Motivation

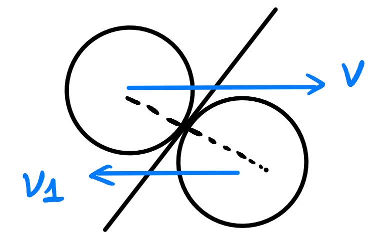
- Consider Boltzmann equation:

$$\partial_t u + v\cdot\nabla_x u = Q(u,u)$$

$$Q(f,g)(v) = \int_{\mathbb{R}^3}\int_{S^2}\left[f(v')g(v_1') - f(v)g(v_1)\right]B(|v-v_1|,\omega)\,d\omega\,dv_1$$

$\hookrightarrow \phi(|v-v_1|)\,b(\cos\langle v-v_1,\omega\rangle)$

$b(\cos\theta) \approx |\cos\theta|^{-1-\alpha}, \quad \alpha\in(0,2).$



- $$Q(f,g) \approx \int_{\mathbb{R}^3} \frac{f(v+\omega)-f(v)}{|\omega|^{d+\alpha}} K_g(v,\omega)\,d\omega + H_g(f)$$ [Chen-Zhang 2018, JMPA].

$$\Delta^{\frac{\alpha}{2}} f := -(-\Delta)^{\frac{\alpha}{2}} f = \mathrm{p.v.}\int_{\mathbb{R}^3} \frac{f(v+\omega)-f(v)}{|\omega|^{d+\alpha}}\,d\omega \rightsquigarrow \alpha\text{-stable process.}$$

Tanaka: Probabilistic treatment of the Boltzmann equation of maxwellian molecules. PTRF. 1978.

Mischler-Mouhot: Kac's program in kinetic theory. Invent. Math. 2013.

Rüdiger-Sundar: Identification and existence of Boltzmann processes. 2024

Lévy process

- $L = (L_t)_{t \geq 0}$ $\begin{cases} L_0 = 0, \\ L_t - L_s \perp\!\!\!\perp \sigma(L_u, u \leq s), \ L_t - L_s \overset{d}{=} L_{t-s}, \quad 0 \leq s < t \\ \lim\limits_{s \to t} P(|L_t - L_s| > \varepsilon) = 0 \end{cases}$

  $L_t = (L_t^1, \cdots, L_t^d) \in \mathbb{R}^d$

- Lévy-Khintchine formula: $E e^{i\xi L_t} = e^{-t\phi(\xi)}$.

  $\int_{\mathbb{R}^d} 1 \wedge |z|^2 \nu(dz) < \infty$

$$\phi(\xi) = \frac{1}{2} a^2 |\xi|^2 + b \cdot \xi + \int_{\mathbb{R}^d} \left[e^{i\xi z} - 1 - 1_{\{|z| \leq 1\}}(i\xi z) \right] \nu(dz)$$

- Lévy-Itô's decomposition:

  $E N(t, A) = t \nu(A)$

$$L_t = a W_t + bt + \int_{|z| \leq 1} z \tilde{N}(t, dz) + \int_{|z| > 1} z N(t, dz)$$

$$N(t, A) := \sum_{0 \leq s \leq t} 1_{\{\Delta L_s \in A\}} \qquad \Delta L_t = L_t - L_{t-}$$

α-stable process

- $L_{ct} \overset{d}{=} c^{\frac{1}{\alpha}} L_t \quad \alpha \in (0,2]$

- $\nu(A) = \int_0^\infty \frac{1}{r^{1+\alpha}} \int_{S^{d-1}} 1_A(r\omega)\Sigma(d\omega)\,dr$

  $\phi(\xi) = |\xi|^\alpha$

  $\Sigma(d\omega) = d\omega \rightsquigarrow \nu(dz) = C_{\alpha,d}|z|^{-d-\alpha}dz$

  $\hookrightarrow \mathcal{L}f = \Delta^{\frac{\alpha}{2}} f$

  $\mathcal{L}f(x) = \int_{\mathbb{R}^d} [f(x+z) - f(x)]\,\nu(dz)$

- $\alpha = 2 \Leftrightarrow L_t = W_t$

- $L_t = (L_t^1, \cdots, L_t^d)$, $\{L_t^i\}_{i=1}^d$ is a family of i.i.d. 1-dim α-stable processes

  $\Sigma(d\omega) = \sum_{i=1}^d \delta_{\omega_i}(d\omega)$, $\nu(dz) = \sum_{i=1}^d |z_i|^{-1-\alpha} \delta_0(dz_1)\cdots\delta_0(dz_{i-1})\,dz_i\,\delta_0(dz_{i+1})\cdots\delta_0(dz_d)$

  $\hookrightarrow \phi(\xi) = \sum_{i=1}^d |\xi_i|^\alpha$

Our Model

- $$dX_t = (K * \mu_t)(X_t)\,dt + dL_t,$$
  where μ_t is the time marginal law of the solution.
  - distributional-dependent SDE (DDSDE)
  - mean-field SDE
  - McKean-Vlasov SDE
- Fokker-Planck equation:
  $$\partial_t \rho = \Delta^{\frac{\alpha}{2}} \rho - \mathrm{div}((K * \rho) \cdot \rho).$$

$$\begin{cases} \alpha < 1 : & \text{supercritical} \\ \alpha = 1 : & \text{critical} \\ \alpha > 1 : & \text{subcritical} \end{cases}$$

Assume:

$$\inf_{\theta \in S^{d-1}} \int_{S^{d-1}} |\langle \theta, \omega \rangle| \Sigma(d\omega) > 0.$$

$$\left(\begin{array}{ll} \partial_t \rho = \Delta^{\frac{\alpha}{2}} \rho & \|\rho_t\|_{C^\beta} \lesssim t^{-\frac{\beta+\gamma}{\alpha}} \|\rho_0\|_{C^{-\gamma}} \\ \partial_t \rho = a \cdot \nabla \rho & \rho_t(x) = \rho_0(x - at) \\ & \rho_0 \in C^{-\gamma} \Leftrightarrow \rho_t \in C^{-\gamma} \end{array} \right)$$

Well-posedness

$$dX_t = K(X_t)dt + dL_t$$

- (Tanaka-Tsuchiya-Watanabe 1974, JMKU) $d=1$, $K \in C^\beta$ $\begin{cases} \beta \geq 0, \text{ strong } (\alpha \geq 1) \\ \beta > 1-\alpha, \text{ weak } (\alpha \in (0,1)) \end{cases}$

  $\alpha < 1$, $\beta \in (0, 1-\alpha) \Rightarrow$ ill-posed.
- (Priola 2012, Osaka JM) $K \in C^\beta$ $\beta > 1 - \frac{\alpha}{2}$, strong $(\alpha > 1)$
- (Chen-Song-Zhang 2018, RMI) $K \in C^\beta$ $\begin{cases} \beta > 1 - \frac{\alpha}{2}, \text{ strong} \\ \beta > (1-\alpha) \vee 0, \text{ weak} \end{cases}$ $(\alpha > \frac{2}{3})$
- (Chen-Zhang-Zhao 2021, TAMS) Multiplicative noise $(\alpha > 0)$

$\alpha > 1$, $K \in C^\beta$, $\beta \in (\frac{1-\alpha}{2}, 0)$:

- (Athreya-Butkovsky-Mytnik 2020, AOP) $d=1$, strong
- (Ling-Zhao 2022, JDE) $\Sigma(dw) = dw$, weak
- (H.-Wu 2023) weak

  } Multiplicative noise
- Xie-Zhang, Chaudru de Raynal-Menozzi-Priola, H.-Wang-Wu, Zhao, Chaudru de Raynal-Menozzi

  ...

Well-posedness

$$dX_t = (K * \mu_t)(X_t)\,dt + dL_t$$

$$\Sigma(dw) = dw; \quad K \in C^\beta \begin{cases} \beta > (1-\alpha) \vee 0, & \text{weak} \\ \beta > 1 - \frac{\alpha}{2}, & \text{strong} \end{cases}$$

- (Frikha-Konakov-Menozzi 2021, DCDS): $\alpha > \frac{2}{3}$
- (Deng-Huang 2024): Multiplicative noise $\alpha > \frac{1}{2}$

$\alpha > 1$: Huang-Yang 2021, NA.
Chaudru de Raynal-Jabir-Menozzi 2022, 2023
H.-Röckner-Zhang 2024+, AOP, ...

Theorem 1. (Well-posedness)

Assume $\alpha \in (0,1)$,

$$\inf_{\theta \in S^{d-1}} \int_{S^{d-1}} |\langle \theta, \omega \rangle| \, \Sigma(d\omega) > 0 .$$

Let $K \in C^{\beta}$.

$$\begin{cases} \beta > 1-\alpha \Rightarrow \text{weak well-posed} \\ \beta > 1-\frac{\alpha}{2} \Rightarrow \text{strong well-posed.} \end{cases}$$

Approximation

- (N-particle system $\mathcal{P}_N X := (X^{N,1}, \cdots, X^{N,N})$)

$$dX_t^{N,i} = \frac{1}{N}\sum_{j=1}^{N} K(X_t^{N,i} - X_t^{N,j})\,dt + dL_t^i, \quad i = 1, 2, \cdots, N,$$

where $\{L_t^i\}_{i=1}^{\infty}$ is a family of i.i.d. α-stable process.

- (Euler scheme $\mathcal{E}_h X := X^h$)

$$X_t^h = X_{kh}^h + (t-kh)(K * \mu_{kh}^h)(X_{kh}^h) + L_t - L_{kh}, \quad t \in (kh, (k+1)h], \; k = 1, 2, \cdots, [\tfrac{T}{h}]$$

where μ_t^h is the time marginal law of $(X_t^h)_{t \in [0,T]}$.

Approximation

- $(X^{N,h,1}, \cdots, X^{N,h,N}) = \mathcal{E}_h \mathcal{P}_N X = \mathcal{P}_N \mathcal{E}_h X$:

$$X_t^{N,h,i} = X_{kh}^{N,h,i} + (t-kh)\frac{1}{N}\sum_{j=1}^{N} K(X_{kh}^{N,h,i} - X_{kh}^{N,h,j}) + L_t - L_{kh}, \quad t \in (kh,(k+1)h]$$

$\mathcal{P}_N X$:

$$X_t^{N,i} = X_0^i + \int_0^t \frac{1}{N}\sum_{j=1}^{N} K(X_s^{N,i} - X_s^{N,j})\,ds + L_t^i$$

$\mathcal{E}_h$. :

$$X_t^h = X_{kh}^h + (t-kh)(K * \mu_{kh}^h)(X_{kh}^h) + L_t - L_{kh}$$

X :

$$X_t = X_0 + \int_0^t (K * \mu_s)(X_s)\,ds + L_t$$

Aim

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{E}_h \mathcal{P}_N X \arrow[r, "\text{PoC for Euler scheme}"] \arrow[d, "\text{Euler approximation for non-distri dependent SDEs}"'] & \mathcal{E}_h X \arrow[d, "\text{Euler approximation for DDSDES.}"] \\
\mathcal{P}_N X \arrow[r, "\text{Propagation of chaos}"] & X
\end{tikzcd}
$$

Well-known results

- $\mathcal{E}_h \mathcal{P}_N X \to \mathcal{P}_N X$

Euler approximation for (non distributional-dependent) SDE driven by α-stable processes

$(\alpha > 1)$ [Mikulevičius-Xu 2018, Stochastics], [Huang-Liao 2018, SAA], [Huang-Suo-Yuan 2023, NA]

[Chen-Deng-Schilling-Xu 2023, SPA (invariant measure)]

$(\alpha > \frac{2}{3})$ [Butkovsky-Dareiotis-Gerencsér 2022 (new for Lipschitz drifts)]

$(\alpha > 0)$ [Li-Zhao 2024, SPL] strong convergence

$(\alpha > 1)$ [Fitoussi-Jourdain-Menozzi 2024], [Song-H. 2024] $\longrightarrow$ weak convergence

- $\mathcal{P}_N X \to X$

propagation of chaos for DDSDE driven by α-stable processes

(Lipschitz kernel) [Graham 1992, SPA], [Jourdain-Méléard-Woyczynski 2008, ALEA]

(Hölder in space & $\alpha > 1$) [Cavallazzi 2022]

Main results

$$\begin{tikzcd} \mathcal{E}_h \mathcal{P}_N X \arrow[r, "①"] \arrow[d] & \mathcal{E}_h X \arrow[d, "②"] \\ \mathcal{P}_N X \arrow[r, "③"] & X \end{tikzcd}$$

Assume $\alpha \in (0,1)$

- Let $\beta \in (0,1)$. Then

$$\lim_{N\to\infty} E\Big[\sup_{t\in[0,T]} |X_t^{N,h,i} - X_t^{h,i}|\Big] = 0 \rightsquigarrow ①$$

- Case 1: $\beta \in (1-\alpha, 1) \longleftrightarrow$ weak case

  - $$\sup_{t\in[0,T]} |E\varphi(X_t^h) - E\varphi(X_t)| \lesssim \|\varphi\|_{C^\beta} h^{\frac{\alpha\wedge\beta}{\alpha}} \rightsquigarrow ②$$

  - if $\mathcal{L}(X_0^{N,1}, \cdots, X_0^{N,k}) \to (\mathcal{L}(X_0))^{\otimes k}$, then $\mathcal{L}(X_t^{N,1}, \cdots, X_t^{N,k}) \to (\mathcal{L}(X_t))^{\otimes k}$, $\forall k$, as $N \to \infty \rightsquigarrow ③$

- Case 2: $\beta \in (1-\frac{\alpha}{2}, 1) \longleftrightarrow$ strong case

  - $$E\Big[\sup_{t\in[0,T]} |X_t^h - X_t|^p\Big] \lesssim h^{\frac{\beta p}{\alpha}\wedge 1} \rightsquigarrow ②$$

  - Assume $(\mathcal{P}_N X)_0 = (X_0^1, \cdots, X_0^N)$ is i.i.d.

    Then $$\lim_{N\to\infty} E\Big[\sup_{t\in[0,T]} |X_t^{N,i} - X_t|\Big] = 0 \rightsquigarrow ③.$$

Draft for proof

- Probability measure space : $\mathcal{P}(\mathbb{R}^d)$ [Finite measure space $M(\mathbb{R}^d)$]

- Measure distance: given $\beta \in (0,1)$

$$\|\mu - \nu\|_{\beta;var} := \sup_{\varphi \in C^\beta} \left| \int_{\mathbb{R}^d} \varphi(x)(d\mu - d\nu)(x) \right|$$

$$\|\varphi\|_{C^\beta} := \|\varphi\|_\infty + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\beta}$$

  - $(\mathcal{P}(\mathbb{R}^d), \|\cdot\|_{\beta;var})$ is complete;

    $(M(\mathbb{R}^d), \|\cdot\|_{\beta;var})$ is not complete.

- Consider the following two SDEs:

$$dX_t^i = b^i(t, X_t)\,dt + dL_t, \qquad \alpha \in (0,1), \qquad i = 1,2.$$

- (Stability estimate)

$$\| \mathcal{L}(X_t^1) - \mathcal{L}(X_t^2) \|_{\beta;\mathrm{var}} \lesssim \int_0^t (t-s)^{-\frac{1-\beta}{\alpha}} \| b^1(r) - b^2(r) \|_\infty \, dr$$

proof: consider $\partial_t u + \Delta^{\frac{\alpha}{2}} u + b^1 \cdot \nabla u = 0$, $u(t) = \varphi \in C^\beta$

Itô's formula

$$\Rightarrow \quad E\varphi(X_t^1) - E\varphi(X_t^2) = E\int_0^t (b^1 - b^2)(s) \cdot \nabla u(t-s)(X_s^2)\, ds$$

$$\|\nabla u(t)\|_{L^\infty} \lesssim t^{-\frac{1-\beta}{\alpha}} \|\varphi\|_{C^\beta}$$

$$dX_t = b(t, X_t, \mu_{X_t})\,dt + dL_t\,, \quad \alpha \in (0,1)$$

- Assumption: $\exists\, \beta \in (1-\alpha, 1)$

$$\left.\begin{aligned} &\sup_{t,x} |b(t,x,\mu) - b(t,x,\nu)| \le C \|\mu - \nu\|_{\beta;var}. \\ &\sup_{t,\mu} \|b(t,\cdot,\mu)\|_{C^\beta} < \infty. \end{aligned}\right\} \Rightarrow \text{well-posedness.}$$

- Picard iteration

$$dX_t^{n+1} = b(t, X_t^{n+1}, \mu_{X_t^n})\,dt + dL_t$$

stability

$$\Rightarrow \sup_{m \ge n} \sup_{t \in [0,T]} \|\mathcal{L}(X_t^n) - \mathcal{L}(X_t^m)\|_{\beta;var} \to 0, \quad \text{as } n \to \infty.$$

$\Rightarrow$ existence & uniqueness

stability

Proof for Euler approximation

- Recall: $dX_t^h = b(t, X^h_{\pi_h(t)}, \mu^h_{\pi_h(t)})dt + dL_t$ $\quad \pi_h(t) = [\frac{t}{h}]h$, $h << 1$.

  $dX_t = b(t, X_t, \mu_{X_t})dt + dL_t$

- $\partial_t u + \Delta^{\frac{\alpha}{2}} u + b \cdot \nabla u = 0$, $\quad u(t) = 0$

  $\hookrightarrow$ $E\varphi(X_t) - E\varphi(X_t^h) = E\int_0^t (b(s, X_s^h, \mu_s) - b(s, X^h_{\pi_h(s)}, \mu^h_{\pi_h(s)})) \cdot \nabla u(t-s, X_s^h) ds$.

Fokker-Planck-Kolmogorov equation

$$\partial_t u = \Delta^{\frac{\alpha}{2}} u - \lambda u + b\cdot\nabla u + f, \quad u_0 = \varphi$$

Theorem: Let $\alpha, \beta \in (0,1)$ with $\alpha+\beta>1$. Assume $b \in C^\beta$.

For $\forall\ \gamma \in [0,\alpha)$, there is a constant $C = C(d, T, \alpha, \beta, \gamma, \|b\|_{C^\beta})$, such that for $\forall\ \lambda \geq 0$, $t \in (0,T]$,

$$\|u(t)\|_{C^{\beta+\gamma}} \leq C\left(t^{-\frac{\gamma}{\alpha}} \|u(0)\|_{C^\beta} + (1+\lambda)^{-\frac{\alpha-\gamma}{\alpha}} \|f\|_{L^\infty_T C^\beta} \right).$$

Difficulty: $\alpha < 1$. $\left(b \in C^\beta \xrightarrow{\text{Schauder}} u \in C^{\alpha+\beta} \rightarrow b\cdot\nabla u \in C^{\beta+(\alpha-1)\,(<\beta)} \right)$

Method: Characteristic line $\alpha > \frac{1}{2}$ [H.-Wu-Zhang, 2020, JMPA]. [H.-Wang-Wu, 2023, PA]

Energy estimate $\alpha > 0$ [Chen-Zhang-Zhao 2021, TAMS], [Song-Xie, 2023, JDE].

Propagation of Chaos

- $\mathcal{P}_N X \to X$: Martingale method

  $\hookrightarrow \mathcal{L}\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X^{N,i}(\omega)}\right) \to \delta_\mu$

  (μ is the unique martingal solution to

  $dX_t = (K * \mu_{X_t})(X_t)\,dt + dL_t$)

- $\mathcal{E}_h \mathcal{P}_N X \to \mathcal{E}_h X$: induction method in [Zhang, 2019, EJP]

Thank you!

谢谢！